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Hence,

$$\frac{\sin B}{\sin A} = \frac{\sin (kA + B)}{\sin (kB + A)}.$$

By composition and division,

$$\frac{\sin B - \sin A}{\sin B + \sin A} = \frac{\sin (kA + B) - \sin (kB + A)}{\sin (kA + B) + \sin (kB + A)};$$

whence,

$$\frac{\tan \frac{1}{2}(B - A)}{\tan \frac{1}{2}(B + A)} = \frac{\tan \frac{1-k}{2}(B - A)}{\tan \frac{1+k}{2}(B + A)}; \quad \text{or} \quad \frac{\tan \frac{1}{2}(B - A)}{\tan \frac{1-k}{2}(B - A)} = \frac{\tan \frac{1}{2}(B + A)}{\tan \frac{1+k}{2}(B + A)}.$$

Since $0 < k < 1$ and the tangent is an increasing function, it follows that if B and A were unequal, the last relation would be untrue, for then the left-hand fraction would be greater than unity, and the right-hand fraction would be less than unity. Hence, $B = A$, and the triangle is isosceles.

488. Proposed by ROGER A. JOHNSON, Western Reserve University.

If triangles are constructed on a given base, having the radii of the incircle and circumcircle in a constant ratio, determine the locus of the vertex (necessarily the constant ratio is not greater than $\frac{1}{2}$).

SOLUTION BY PAUL CAPRON, U. S. Naval Academy.

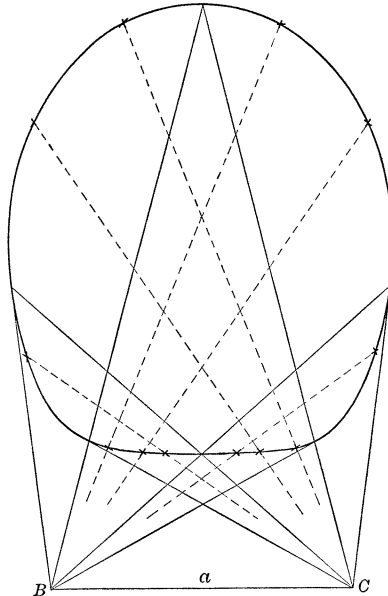
Let the constant ratio be $n = 2m$, the given base a , with its left end at B , its right end at C ; and let the sides a, b, c of any one of the triangles be opposite the vertices A, B, C respectively.

Then

$$n = 2m = \frac{ac \sin B}{a + b + c} \div \frac{b}{2 \sin B} = \frac{2ac \sin^2 B}{b(a + b + c)}.$$

Let $c = r, B = \theta = 2\phi$; then $b^2 = a^2 + r^2 - 2ar \cos \theta = (a + r)^2 - 4ar \cos^2 \phi$. On substituting these values, we have, for the locus of A :

$$(1) \quad m(a + r) \sqrt{(a + r)^2 - 4ar \cos^2 \phi} = 4ar \cos^2 \phi (m + \sin^2 \phi) - m(a + r)^2.$$



Rationalizing:

$$(2) \quad r \cos^2 \phi [(m^2 + 2m \sin^2 \phi)(a + r)^2 - 4a \cos^2 \phi (m + \sin^2 \phi)^2 r] = 0.$$

As $A \doteq B$, $m \doteq 0$; except in this trivial case, $r = 0$ may be discarded. Solving (2) for $(r + a)$ we find,

$$(3) \quad r + a = \frac{2a \cos^2 \phi (m + \sin^2 \phi)^2}{m^2 + 2m \sin^2 \phi} \pm \frac{2a \sin \phi \cos \phi}{m^2 + 2m \sin^2 \phi} (m + \sin^2 \phi) \sqrt{\sin^2 \phi - (m + \sin^2 \phi)^2}.$$

Replacing m and ϕ by $\frac{1}{2}n$ and $\frac{1}{2}\theta$,

$$(4) \quad r = a \cos \theta + \frac{a \sin \theta}{n^2 + 2n(1 - \cos \theta)} [\sin \theta (1 - \cos \theta) \pm (n + 1 - \cos \theta) \sqrt{1 - 2n - (\cos \theta - n)^2}],$$

or

$$(4)' \quad r = a \cos \theta + \frac{a \sin^2 \theta \text{ vers } \theta}{n^2 + 2n \text{ vers } \theta} \pm \frac{a \sin \theta (n + \text{vers } \theta)}{n^2 + 2n \text{ vers } \theta} \sqrt{\sin^2 \theta - 2n \text{ vers } \theta - n^2},$$

where $\text{vers } \theta = 1 - \cos \theta$.

When $\theta = 0$, $r = a$, and when $\theta = \pi$, $r = -a$; this point C must be discarded from the locus, just as B was.

Aside from this, r is real when and only when

$$(5) \quad n - \sqrt{1 - 2n} \leq \cos \theta \leq n + \sqrt{1 - 2n},$$

or

$$1 - n + \sqrt{1 - 2n} \geq \text{vers } \theta \geq 1 - n - \sqrt{1 - 2n}.$$

(6) When θ has either of the limiting values given by (5), then $r = a$ and the third sides of these isosceles triangles are

$$a(1 \pm \sqrt{1 - 2n}).$$

Considerations of symmetry show that the high and low points of the curve occur where $r = a/2 \sec \theta$. Substituting this value in (2), we have:

$$m(m + 1 - \cos \theta)(2 \cos \theta + 1)^2 - \cos \theta(1 + \cos \theta)(2m + 1 - \cos \theta)^2 = 0;$$

whence

$$(\cos^2 \theta - \cos \theta + m)(\cos^2 \theta - (m + 1)) = 0,$$

and

$$(7) \quad \cos \theta = \pm \frac{1}{2} \sqrt{4 + 2n},$$

or

$$(8) \quad \cos \theta = \frac{1}{2}(1 \mp \sqrt{1 - 2n}).$$

The values given by (7) are without the limits set in (5); those given by (8) are within these limits. From (8),

$$(9) \quad r = \frac{a}{2n} (1 \pm \sqrt{1 - 2n}),$$

the high and low points; or

$$(10) \quad r = a(1 \mp \sqrt{1 - 2n}),$$

giving two other points.

The curve consists of two ovals, symmetrically situated with regard to $\theta = 0$, the common base of the triangles, and each oval is symmetrical with regard to $2r = a \sec \theta$, the mid-perpendicular to this base. In addition to the limiting tangents from B , given by (6), there is a symmetrically situated pair from C ; these, however, are tangent at the points given by (10). Thus in each oval we can readily construct six points, with the tangent at each. These points determine the isosceles triangles satisfying the conditions of the problem. The six triangles form two sets of three similar triangles.

The accompanying figure shows one of the ovals, constructed for $n = 3/8$.

The values used are:

$\cos \theta$	$r = c$	
$-\frac{1}{8}$	a	$(b = \frac{3}{2}a)$
$\frac{1}{4}$	$\begin{cases} 2a \\ \frac{1}{2}a \end{cases}$	$\begin{matrix} \text{High Point } (b = 2a) \\ (b = a) \end{matrix}$

$$\begin{array}{ll}
\frac{3}{8} & \left\{ \frac{a}{39} (49 \pm 4\sqrt{55}) = 2.017a \text{ or } 0.496a \right. \\
\frac{9}{16} & \left\{ \frac{a}{384} (461 \pm 13\sqrt{385}) = 1.865a \text{ or } 0.536a \right. \\
\frac{3}{4} & \left\{ \begin{array}{ll} \frac{3}{4}a & \text{Low Point } (b = \frac{3}{4}a) \\ \frac{3}{4}a & (b = a) \end{array} \right. \\
\frac{13}{16} & \left\{ \frac{a}{128} (133 \pm \sqrt{1305}) = 1.221a \text{ or } 0.757a \right. \\
\frac{7}{8} & a \quad (b = \frac{1}{2}a)
\end{array}$$

CALCULUS.

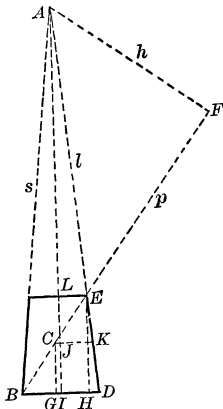
407. Proposed by PAUL CAPRON, Annapolis, Maryland.

A coffee pot in the form of a conical frustum, 10 inches high, with a lower base 8 inches in diameter and an upper base 6 inches in diameter, is held on a slant so that the lower base is barely covered by the coffee within, and the upper base is barely uncovered. How much coffee does the pot contain?

I. SOLUTION BY H. S. UHLER, Yale University.

The following solution may be of interest because it is based exclusively on theorems of elementary geometry. The diagram (drawn to scale) represents a plane section passing through the axis \overline{IA} of the frustum $BDEL$ and the major axis \overline{BE} of the elliptical free-surface of the coffee. The required volume will be gotten by subtracting the volume of the oblique cone ABE from that of the right cone ABD . We are given that $\overline{IL} = 10$ in., $\overline{ID} = 4$ in., and $\overline{LE} = 3$ in.

From the similar triangles AID and ALE , $\overline{AI} : \overline{ID} = \overline{AL} : \overline{LE}$ or $\overline{AI} : 4 = (\overline{AI} - 10) : 3$, hence $\overline{AI} = 40$ in. Hence, the volume of the right cone equals $(\frac{4}{3}\pi) \text{ cu. in.}$



The altitude of the oblique cone may be obtained from the right triangles ABF and AEF , for, $s^2 = (2a + p)^2 + h^2$ and $l^2 = p^2 + h^2$, where a denotes the semi-major axis of the ellipse. That $a = 0.5\sqrt{149}$ may be seen at once from the right triangle BHE since $\overline{BH} = 7$ and $\overline{HE} = 10$. $s^2 = (\overline{BI})^2 + (\overline{AI})^2 = 1616$ and $l^2 = (\overline{LE})^2 + (\overline{LA})^2 = 909$. Elimination of p from the twoliteral equations gives $h = 240/\sqrt{149}$ in. It remains to find the semi-minor axis b of the ellipse. This axis will pass through C , the middle point of \overline{BE} , it will be perpendicular to the plane of the diagram, and it will be a chord of the circle whose radius is \overline{JK} and whose plane is parallel to the bases of the given frustum. Since C bisects \overline{BE} and \overline{CK} is parallel to \overline{BD} , $\overline{BG} = 0.5\overline{BH} = 3.5$. $\overline{CJ} = \overline{BI} - \overline{BG} = 4 - 3.5 = 0.5$. $\overline{JK} = 0.5(\overline{ID} + \overline{LE}) = 3.5$. Since the minor axis of the ellipse constitutes the chord of a circle of radius 3.5 in. and is at a distance 0.5 in. from the center it follows that $b^2 = (3.5 - 0.5)(3.5 + 0.5) = 12$. Hence, $b = 2\sqrt{3}$ in. The area of the ellipse $= \pi ab = \pi\sqrt{3} \times 149$; hence, the volume of the oblique cone equals

$$\frac{1}{3}(\pi\sqrt{3} \times 149) \left(\frac{240}{\sqrt{149}} \right) = 80\pi\sqrt{3} \text{ cu. in.}$$

Consequently, the volume of the coffee equals

$$\frac{80\pi}{3} (8 - 3\sqrt{3}) \text{ cu. in.} \doteq 234.894,585,349,6 \text{ cu. in.}$$